

On Chern-Simons Couplings at order $O(\alpha'^2)$

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Abstract

Using the explicit string scattering calculation and the linear T-dual ward identity, we evaluate the string theory disc amplitude of one Ramond-Ramond field $C^{(p+1)}$ and two Neveu-Schwarz B-fields in the presence of a single D_p -brane in type *IIB* string theory. From this amplitude we extract the $O(\alpha'^2)$ (or equivalently four-derivative) part of the D_p -brane couplings involving these fields.

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1 Introduction

Achieving a detailed understanding of string theory vacua plays a very important role in the relationship between this theory with real-world physics. One of the most ingredients of vacuum is D -brane (localized objects on which strings can end). D -branes are non-perturbative excitations in the theory which are localized to a sub-manifold of the ten-dimensional space-time. We will be considering the stable D -branes that carry the RR charges. There are many degrees of freedom localized on the D -brane's worldvolume.

Most of D -brane's researches being done focuses on their low-energy dynamics, and so the effects of higher derivative corrections to the string theory action are often ignored. However, it has been shown that the corrections of high order, has a very important role in establishing the consistency, or inconsistency, of some solutions. The investigation of these solutions demonstrate that the higher derivative corrections to the D -brane actions can also have a major role in determining the consistency of string compactifications. Regardless of these corrections, understanding the proper structure of the string vacua is not possible [2, 4, 5].

BPS D_p -branes in type II string theories (p even(odd) in $IIA(IIB)$ theory) are stable solitons which break half of the space-time supersymmetries and their dynamics are properly described in field theory by Dirac-Born-Infeld (DBI) action and Chern-Simons action (CS) [6, 7].

The DBI action describes the dynamics of the brane in the presence of $NSNS$ background fields. The CS part describes the coupling of D -branes to the RR potential. For constant fields it is given by [8, 9]

$$S_{CS} = T_p \int_{M^{p+1}} e^B C \quad (1)$$

where M^{p+1} represents the world volume of the D_p -brane, C is the sum over all RR potential forms, *i.e.*, $C = \sum_{n=0}^8 C^{(n)}$, and the multiplication rule is wedge product. The abelian gauge field can be added to the action as $B \rightarrow B + 2\pi\alpha' f$. Curvature correction to this action has been found in [10, 11, 12] by requiring that the chiral anomaly on the world volume of intersecting D -branes cancels the anomalous variation of the CS action.

Additional linear couplings to the $NSNS$ fields were found by studying the S-matrix element of one RR and one $NSNS$ vertex operator at order $O(\alpha'^2)$ [14]. These couplings take the following form [16]:

$$\begin{aligned} \pi^2 \alpha'^2 T_p \int d^{p+1}x \epsilon^{a_0 \dots a_p} & \left(\frac{1}{2!(p-1)!} [F_{ia_2 \dots a_p, a}^{(p)} H_{a_0 a_1}{}^{a, i} - F_{aa_2 \dots a_p, i}^{(p)} H_{a_0 a_1}{}^{i, a}] \right. \\ & + \frac{2}{p!} \left[\frac{1}{2!} F_{ia_1 \dots a_{p-1}, a}^{(p+2)} \mathcal{R}^a{}_{a_0}{}^{ij} - \frac{1}{p+1} F_{a_0 \dots a_{p-1}, i}^{(p+2)} (\hat{\mathcal{R}}^{ij} - \phi^{ij}) \right] \\ & \left. - \frac{1}{3!(p+1)!} F_{ia_0 \dots a_{p-1}, a}^{(p+4)} H^{ijk, a} \right) \end{aligned} \quad (2)$$

where the commas are used to denote partial differentiation, \mathcal{R} is the linearized curvature tensor and $F^{(n)} = dC^{(n-1)}$. The above couplings are invariant under the linear T -dual ward identity corresponding to $NSNS$ gauge transformations¹.

These quadratic couplings can be extended to higher order terms by making them covariant under the coordinate transformations and invariant under the following RR gauge transformations:

$$\delta C = d\Lambda + H \wedge \Lambda \quad (3)$$

where H is the field strength of B , *i.e.*, $H = dB$, and $\Lambda = \sum_{n=0}^7 \Lambda^{(n)}$. That is, the partial derivatives in these couplings should be replaced by the covariant derivatives, the closed string tensors should be extended to the pull-back of the bulk fields onto the world-volume of D -brane, the linear RR gauge field strength should be extended to the nonlinear field strength $\tilde{F}^{(n)} = dC^{(n-1)} + H \wedge C^{(n-3)}$, and the linear curvature tensor should be extended to the nonlinear curvature tensor. These couplings for $\mathcal{C}^{(p+1)}$ are[15]:

$$\begin{aligned} \frac{T_p}{(p-1)!} \int d^{p+1}x \epsilon^{a_0 \dots a_{p-2} bd} \mathcal{C}_{a_0 \dots a_{p-2}}^{(p+1)}{}^{ij} & \left[\frac{1}{2} H_j{}^{fe}{}_{,b} H_{ied,f} + \frac{1}{4} \mathcal{R}_{bd}{}^{ef} \mathcal{R}_{ijfe} + \mathcal{R}^e{}_{jb}{}^k \mathcal{R}_{eidk} \right. \\ & \left. - \frac{1}{2} H_j{}^{lk}{}_{,b} H_{ikd,l} - \frac{1}{4} \mathcal{R}_{bd}{}^{kl} \mathcal{R}_{ijlk} - \mathcal{R}^k{}_{jb}{}^e \mathcal{R}_{kide} \right] \quad (4) \end{aligned}$$

In the present paper, we are interested in generalizing the CS action to take into account some other couplings of $C^{(p+1)} \wedge H \wedge H$. To compute the terms of interest, we must evaluate scattering amplitudes in which various closed string fields interact with a D-brane. We will restrict ourselves to tree-level computations, so the relevant amplitudes are given by insertions of multiple closed string vertex operators on a world-sheet with the topology of a disc. From these S-matrix elements, we conjecture an extension for the CS action which includes these couplings as well. In fact, we will present some four-derivative corrections involving a RR potential of degree $p+1$, and two $NSNS$ fields.

The outline of this paper is as follows. In section 2 we start by writing the S-matrix elements of one RR and two $NSNS$ vertex operators. From this approach, we find the S-matrix elements for the RR potential with degree $(p+1)$ that carry four and three transvers indices. we check the consistency of these elements with the results in [21] where the relevant amplitudes found by using of T-dual ward identity. In section 3, by applying the explicit scattering calculation, we drive the amplitude for RR $(p+1)$ -form with one transverse index in terms of the RR structures. Next, in section 4, we calculate the amplitude in low energy limit and find some new couplings.

¹Extension of the couplings in (2) to one RR , one $NSNS$ and one NS open string couplings have been studied in [22].

2 The approach for three closed string amplitude; one RR and two $NSNS$

The explicit calculations of the scattering amplitude of one RR n -form in $(-1/2, -3/2)$ -picture and two $NSNS$ vertex operators in $(0, 0)$ -picture on the world volume of D_p -brane has been studied in [5, 17, 18, 19]. We review this calculations and apply them to find the interesting amplitudes. We work in the RNS world-sheet formalism, with the closed string vertex operators being constructed out of bosons $X^\mu(z), X^\mu(\bar{z})$ and fermions $\psi^\mu(z), \psi^\mu(\bar{z})$ as well as the picture ghosts $\phi(z)$ and $\phi(\bar{z})$. The scattering amplitude is given by the following correlation function:

$$\mathcal{A} \sim \langle V_{RR}^{(-1/2, -3/2)}(\varepsilon_1^{(n)}, p_1) V_{NSNS}^{(0,0)}(\varepsilon_2, p_2) V_{NSNS}^{(0,0)}(\varepsilon_3, p_3) \rangle \quad (5)$$

Using the doubling trick [14] and standard relevant vertex operators [3], explicitly we have

$$\begin{aligned} \mathcal{A} \sim & \frac{1}{2} (H_{1(n)} M_p)^{AB} (\varepsilon_2 \cdot D)_{\mu_3 \mu_4} (\varepsilon_3 \cdot D)_{\mu_5 \mu_6} \int d^2 z_1 d^2 z_2 d^2 z_3 (z_{1\bar{1}})^{-3/4} \\ & \left(\sum_{i=1}^{10} b_i + \sum_{i=1}^6 b'_i \right)_{AB}^{\mu_3 \mu_4 \mu_5 \mu_6} \delta^{p+1} (p_1^a + p_2^a + p_3^a) + (2 \leftrightarrow 3) \end{aligned} \quad (6)$$

where the $NSNS$ vertex operators include polarizations ε_i and momenta p_i , $i = 2, 3$. The RR vertex operator has momentum p_1 and antisymmetric polarization ε_1 . The indices A, B, \dots are the Dirac spinor indices, and

$$\begin{aligned} H_{1(n)} &= \frac{1}{n!} \varepsilon_{1\mu_1 \dots \mu_n} \gamma^{\mu_1} \dots \gamma^{\mu_n} \\ M_p &= \frac{\pm 1}{(p+1)!} \epsilon_{a_0 \dots a_p} \gamma^{a_0} \dots \gamma^{a_p} \end{aligned} \quad (7)$$

where ϵ is the volume $(p+1)$ -form of the D_p -brane. Here the matrix $D_{\mu\nu}$ is a diagonal matrix that agrees with $\eta_{\mu\nu}$ in directions along the brane (Neumann boundary conditions) and with $-\eta_{\mu\nu}$ in directions normal to the brane (Dirichlet boundary conditions). In this notation, $D^{\mu i} = -\delta^{\mu i}$, $D^{\mu a} = \delta^{\mu a}$ and thus $D_{\mu\nu} = V_{\mu\nu} - N_{\mu\nu}$. One has to use the standard world-sheet propagators to write amplitude (6). b_i 's are some correlation functions. These functions include the correlators of X 's, the correlators of two spin operators and the correlation function involving two spin operators and ψ 's. b_i 's have been illustrated in [17, 18]. b'_i 's are also some correlation functions that contribute to the amplitude of our interested cases in this paper

$$\begin{aligned} (b'_{11})_{AB}^{\mu_3 \mu_4 \mu_5 \mu_6} &= (ip_3)_{\beta_1} \langle : S_A : S_B : \psi^{\beta_1} \psi^{\mu_5} : \rangle g_{11}^{\mu_3 \mu_4 \mu_6} \\ (b'_{12})_{AB}^{\mu_3 \mu_4 \mu_5 \mu_6} &= (ip_3 \cdot D)_{\beta_1} \langle : S_A : S_B : \psi^{\beta_1} \psi^{\mu_6} : \rangle g_{12}^{\mu_3 \mu_4 \mu_5} \\ (b'_{13})_{AB}^{\mu_3 \mu_4 \mu_5 \mu_6} &= (ip_2 \cdot D)_{\beta_1} (ip_3)_{\beta_2} \langle : S_A : S_B : \psi^{\beta_1} \psi^{\mu_4} : \psi^{\beta_2} \psi^{\mu_5} : \rangle g_{13}^{\mu_3 \mu_6} \end{aligned}$$

$$\begin{aligned}
(b'_{14})_{AB}^{\mu_3\mu_4\mu_5\mu_6} &= (ip_3)_{\beta_1}(ip_3 \cdot D)_{\beta_2} <: S_A : S_B : \psi^{\beta_1} \psi^{\mu_5} : \psi^{\beta_2} \psi^{\mu_6} :> g_{14}^{\mu_3\mu_4} \\
(b'_{15})_{AB}^{\mu_3\mu_4\mu_5\mu_6} &= (ip_2)_{\beta_1}(ip_3)_{\beta_2}(ip_3 \cdot D)_{\beta_3} <: S_A : S_B : \psi^{\beta_1} \psi^{\mu_3} : \psi^{\beta_2} \psi^{\mu_5} : \psi^{\beta_3} \psi^{\mu_6} :> g_{15}^{\mu_4} \\
(b'_{16})_{AB}^{\mu_3\mu_4\mu_5\mu_6} &= (ip_2 \cdot D)_{\beta_1}(ip_3)_{\beta_2}(ip_3 \cdot D)_{\beta_3} <: S_A : S_B : \psi^{\beta_1} \psi^{\mu_4} : \psi^{\beta_2} \psi^{\mu_5} : \psi^{\beta_3} \psi^{\mu_6} :> g_{16}^{\mu_3}
\end{aligned}$$

where g 's are the correlators of X 's which can easily be performed using the standard world-sheet propagators, and the correlator of ψ can be calculated using the Wick-like rule. Respecting the following relation between b_i and b'_i in [17, 18], it could be found that the contribution of the above correlators appear as a overall factor of 2.

$$b'_{11} = b_2, \quad b'_{12} = b_3, \quad b'_{13} = b_6, \quad b'_{14} = b_4, \quad b'_{15} = b_8, \quad b'_{16} = b_9.$$

One can find that the amplitude involves the following trace that appears by combining the gamma matrices in amplitude (6) and the Wick-like rule gamma matrices [17]:

$$T(n, p, m) = \frac{1}{n!(p+1)!} \varepsilon_{1\nu_1 \dots \nu_n} \epsilon_{a_0 \dots a_p} A_{[\alpha_1 \dots \alpha_m]} \text{Tr}(\gamma^{\nu_1} \dots \gamma^{\nu_n} \gamma^{a_0} \dots \gamma^{a_p} \gamma^{\alpha_1} \dots \gamma^{\alpha_m}) \quad (8)$$

where $A_{[\alpha_1 \dots \alpha_m]}$ is an antisymmetric combination of the momenta and/or the polarizations of the $NSNS$ states and $m = 0, 2, 4, 6, 8$. The scattering amplitude is non-zero only for the following sequence [17]:

$$n = p - 3, n = p - 1, n = p + 1, n = p + 3, n = p + 5 \quad (9)$$

The explicit calculations in order to find the amplitudes corresponding to the first two terms of this sequence have been done in [5, 17, 18, 19]. As we have illustrated in Table 1, we separate the known RR corresponding amplitudes that has been calculated explicitly in literatures. We are going to continue our efforts to find all RR n-form amplitudes. In this paper, we are interested in the case $n = p + 1$. In this case we consider the RR potentials with degree $n = p + 1$ that carry four, three and two transverse indices. The nonzero trace relation (8) for this case mentions in Table 1.

In the following, we will evaluate the amplitude corresponding to $n = p + 1$ RR potentials. Since the similar calculations for another cases of RR potentials have been performed in the papers mentioned above, thus the summary of results for interesting cases presented here. Let us begin with ε_1^{ijkl} .

2.1 $n = 4$ case

From the relation $n = p + 1$ for this RR potential ε_1^{ijkl} , one gets $p = 3$. The nonzero trace (8) becomes

$$T(4, 3, 8) = 32 \frac{8!}{4!4!} \varepsilon_1^{ijkl} \epsilon^{a_0 a_1 a_2 a_3} A_{[ijkl a_0 a_1 a_2 a_3]} \quad (10)$$

Table 1: The calculated scattering amplitudes of one RR and two $NSNS$ in explicit approach

	transverse indices	nonzero trace	reference
n=p-3	2	T(2, 5, 8)	[17], [5]
n=p-3	1	T(1, 4, 6)	[17], [5]
n=p-3	0	T(0, 3, 4)	[18], [5]
n=p-1	3	T(3, 4, 8)	[19]
n=p-1	2	T(2, 3, 6)	[19]
n=p-1	1	T(1, 2, 4)	[19]
n=p+1	4	T(4, 3, 8)	current paper
n=p+1	3	T(3, 2, 6)	current paper
n=p+1	2	T(2, 1, 4)	current paper

where 32 is the trace of the 32×32 identity matrix. Since $m = 8$, only the ψ correlator in b_{10} have non-zero contribution to the amplitude (5). After mapping the results to disk with unit radius and set [4] $z_1 = 0$ to fix $SL(2, R)$ symmetry, it can be found the X correlator in b_{10} is [17, 18]

$$|z_2|^{2s}|z_3|^{2t} \left(1 - |z_2|^2\right)^p \left(1 - |z_3|^2\right)^q |z_2 - z_3|^{2u} |1 - z_2 \bar{z}_3|^{2v} \equiv K \quad (11)$$

where we have used the following definitions for the Mandelstam variables:

$$\begin{aligned} s &= p_1 \cdot p_2 & ; & \quad t = p_1 \cdot p_3 & ; & \quad u = p_2 \cdot p_3 \\ p &= p_2 \cdot D \cdot p_2 & ; & \quad q = p_3 \cdot D \cdot p_3 & ; & \quad v = p_2 \cdot D \cdot p_3 \end{aligned} \quad (12)$$

Using the fact that the kinematic factor in (10) contract with $\varepsilon_1^{ijkl} \epsilon^{a_0 a_1 a_2 a_3}$, one observes that there are 70 different terms, however, some of them are zero and finally two terms remain independently. Considering the ψ -correlator from Wick-like rule and the above X -correlator in (6), one finds that the scattering amplitude is nonzero only for the $NSNS$ polarization tensors with the same symmetry. This amplitude becomes²:

$$\begin{aligned} \mathcal{A} \sim & T_3 \epsilon_{a_0 \dots a_3} p_2^{a_0} p_3^{a_1} \left(\frac{1}{4} (\varepsilon_2^S \cdot N \cdot \mathcal{E} \cdot N \cdot \varepsilon_3^S)^{a_2 a_3} + \text{Tr}(\mathcal{E} \cdot N \cdot \varepsilon_2^A) (\varepsilon_3^A)^{a_2 a_3} \right) \\ & \times \mathcal{I}_1 \delta(p_1^a + p_2^a + p_3^a) + (2 \leftrightarrow 3) \end{aligned} \quad (13)$$

where T_3 is the tension of D₃-brane and $\varepsilon^A, \varepsilon^S$ are the polarizations of the B-field and graviton respectively. We define $\mathcal{E}^{nm} = (p_2 \cdot N \cdot \varepsilon_1 \cdot N \cdot p_3)^{nm}$ in which n, m are the transvers indices.

²Our conventions set $\alpha' = 2$ in the string theory amplitudes. Our index convention is that the Greek letters (μ, ν, \dots) are the indices of the space-time coordinates, the Latin letters (a, d, c, \dots) are the world-volume indices and the letters (i, j, k, \dots) are the normal bundle indices.

The closed and open string channels appear in the integral \mathcal{I}_1 . The explicit form of this integral and its low energy expansion has been found in [17]. In future section, we will study all integrals that appear in the three closed string amplitudes. The above amplitude has been found by T -duality in [20].

2.2 $n = 3$ case

In this case of $(p+1)$ -form RR potential ε_1^{ijk} that carries three transverse indices, we have $p = 2$. The ψ correlators in $b_8, b_9, b_{10}, b'_{15}, b'_{16}$ have non-zero contribution to the amplitude (5). It could be found that this scattering amplitude is zero when the symmetry of $NSNS$ polarizations are different. So the amplitude is non-zero for two graviton or two B-field vertex operators. The trace relation (8) is non-zero only for $m = 6$. The trace in this case becomes:

$$T(3, 2, 6) = 32 \frac{6!}{3!3!} \varepsilon_1^{ijk} \epsilon^{a_0 a_1 a_2} A_{[ijk a_0 a_1 a_2]} \quad (14)$$

Using the on-shell condition $\varepsilon_2 \cdot p_2 = \varepsilon_3 \cdot p_3 = 0$ and conservation of momentum along the world volume, it could be found that the X correlators corresponding to these b_i 's and b'_i 's. After calculating the sub-amplitudes \mathcal{A}_i in (6) corresponding to $b_8, b_9, b_{10}, b'_{15}, b'_{16}$, one can find that the polarization of RR $(p+1)$ -form with three transverse indices appears in amplitude in the form of following RR structures:

$$\begin{aligned} & Tr(\mathcal{E} \cdot N \cdot \varepsilon^A \cdot N), \quad (p \cdot \varepsilon^S \cdot N \cdot \mathcal{E} \cdot N \cdot \varepsilon^S)^\mu, \quad (\varepsilon^S \cdot N \cdot \mathcal{E} \cdot N \cdot \varepsilon^S)^{\mu\nu}, \quad (\mathcal{E} \cdot N \cdot \varepsilon^S)^\mu \\ & (p \cdot \varepsilon^A \cdot N \cdot \mathcal{E}), \quad (\mathcal{E} \cdot N \cdot \varepsilon^A \cdot \varepsilon^A)^\mu, \quad (\mathcal{E} \cdot N \cdot \varepsilon^S \cdot \varepsilon^S)^\mu \end{aligned}$$

where μ, ν are the world volume indices that contract with the volume $(p+1)$ -form ϵ . In above, we define $\mathcal{E}^{nm} = (p_i \cdot N \cdot \varepsilon_1)^{nm}$, $i = 2, 3$ that appears in the first line and also $\mathcal{E}^n = (p_i \cdot N \cdot \varepsilon_1 \cdot N \cdot p_j)^n$, $i, j = 2, 3$, $i \neq j$ that appears in the other structures. n, m are the transverse indices. $\varepsilon^A, \varepsilon^S$ are the polarizations of the B-field and graviton respectively.

We find the amplitude of two B -fields in terms of the above RR structures:

$$\begin{aligned} \mathcal{A}_A \sim & \epsilon_{a_0 a_1 a_2} \left[Tr(\mathcal{E} \cdot N \cdot \varepsilon_2^A \cdot N) \mathcal{M}_1^{a_0 a_1 a_2} + Tr(\mathcal{E} \cdot N \cdot \varepsilon_3^A \cdot N) \mathcal{M}_2^{a_0 a_1 a_2} \right. \\ & + (p_1 \cdot N \cdot \varepsilon_3^A \cdot N \cdot \mathcal{E}) \mathcal{M}_3^{a_0 a_1 a_2} + (p_2 \cdot N \cdot \varepsilon_3^A \cdot N \cdot \mathcal{E}) \mathcal{M}_4^{a_0 a_1 a_2} \\ & + (p_2 \cdot V \cdot \varepsilon_3^A \cdot N \cdot \mathcal{E}) \mathcal{M}_5^{a_0 a_1 a_2} + (p_3 \cdot V \cdot \varepsilon_2^A \cdot N \cdot \mathcal{E}) \mathcal{M}_6^{a_0 a_1 a_2} \\ & + (p_3 \cdot N \cdot \varepsilon_2^A \cdot N \cdot \mathcal{E}) \mathcal{M}_7^{a_0 a_1 a_2} + (\mathcal{E} \cdot N \cdot \varepsilon_2^A \cdot V \cdot \varepsilon_3^A)^{a_2} \mathcal{M}_8^{a_0 a_1} \\ & \left. + (\mathcal{E} \cdot N \cdot \varepsilon_2^A \cdot N \cdot \varepsilon_3^A)^{a_2} \mathcal{M}_9^{a_0 a_1} \right] + (2 \leftrightarrow 3) \end{aligned} \quad (15)$$

where the \mathcal{E} tensors (contraction of $NSNS$ momenta with the RR polarization tensor) in the above S-matrix elements that appear in the bracket are $\mathcal{E}^{nm} = (p_2 \cdot N \cdot \varepsilon_1)^{nm}$ and

$\mathcal{E}^n = (p_2 \cdot N \cdot \varepsilon_1 \cdot N \cdot p_3)^n$. Thus we have $\mathcal{E}^{nm} = (p_3 \cdot N \cdot \varepsilon_1)^{nm}$ and $\mathcal{E}^n = (p_3 \cdot N \cdot \varepsilon_1 \cdot N \cdot p_2)^n$ in the $(2 \leftrightarrow 3)$. The subscript A refers to the amplitude that involves two antisymmetric $NSNS$ polarization tensors.

T-duality could find the first line of the above antisymmetric amplitude where only one index of the RR potential contracts with the momentum corresponding to $NSNS$ polarizations and the other indices of the RR potential contract with the $NSNS$ polarization.

Since there is no conservation of transverse momentums in (6), the terms including contraction of p_1^i with each of the polarizations, *i.e.*, $p_1 \cdot N \cdot \varepsilon_3$, $p_1 \cdot N \cdot \varepsilon_2$, as well as $p_2 \cdot N \cdot \varepsilon_3$, $p_3 \cdot N \cdot \varepsilon_2$ are independent structures. For the other terms we use the conservation of momentum along the brane, *i.e.*,

$$(p_1 + p_2 + p_3) \cdot V_\mu = 0 \quad (16)$$

to write p_1^a in terms of p_2^a and p_3^a .

The S-matrix elements of two gravitons in terms of the RR structures are as following:

$$\begin{aligned} \mathcal{A}_S \sim \epsilon_{a_0 a_1 a_2} & \left[(p_1 \cdot N \cdot \varepsilon_3^S \cdot N \cdot \mathcal{E} \cdot N \cdot \varepsilon_2^S)^{a_2} \mathcal{N}_1^{a_0 a_1} + (p_2 \cdot N \cdot \varepsilon_3^S \cdot N \cdot \mathcal{E} \cdot N \cdot \varepsilon_2^S)^{a_2} \mathcal{N}_2^{a_0 a_1} \right. \\ & + (p_3 \cdot V \cdot \varepsilon_2^S \cdot N \cdot \mathcal{E} \cdot N \cdot \varepsilon_3^S)^{a_2} \mathcal{N}_3^{a_0 a_1} + (p_2 \cdot V \cdot \varepsilon_3^S \cdot N \cdot \mathcal{E} \cdot N \cdot \varepsilon_2^S)^{a_2} \mathcal{N}_4^{a_0 a_1} \\ & + (p_3 \cdot N \cdot \varepsilon_2^S \cdot N \cdot \mathcal{E} \cdot N \cdot \varepsilon_3^S)^{a_2} \mathcal{N}_5^{a_0 a_1} + (p_3 \cdot V \cdot \varepsilon_3^S \cdot N \cdot \mathcal{E} \cdot N \cdot \varepsilon_2^S)^{a_2} \mathcal{N}_6^{a_0 a_1} \\ & + [(\varepsilon_2^S \cdot N \cdot \mathcal{E} \cdot N \cdot \varepsilon_3^S)^{a_1 a_2} + (2 \leftrightarrow 3)] \mathcal{N}_7^{a_0} + (\mathcal{E} \cdot N \cdot \varepsilon_3^S \cdot N \cdot \varepsilon_2^S)^{a_2} \mathcal{N}_8^{a_0 a_1} \\ & + (\mathcal{E} \cdot N \cdot \varepsilon_3^S \cdot V \cdot \varepsilon_2^S)^{a_2} \mathcal{N}_9^{a_0 a_1} + (\mathcal{E} \cdot N \cdot \varepsilon_3^S)^{a_2} \mathcal{N}_{10}^{a_0 a_1} \\ & \left. + (\mathcal{E} \cdot N \cdot \varepsilon_2^S)^{a_2} \mathcal{N}_{11}^{a_0 a_1} \right] + (2 \leftrightarrow 3) \end{aligned} \quad (17)$$

The subscript S refers to the amplitude that involves two symmetric $NSNS$ polarization tensors. The first three lines and the first term of the fourth line of the above symmetric amplitude could capture by T-duality. One needs the T-dual Ward identity to find the other terms. \mathcal{N}_i ; $i = 1, \dots, 11$ (\mathcal{M}_i ; $i = 1, \dots, 9$) are the combination of momenta, symmetric(antisymmetric) $NSNS$ polarizations and integrals functions:

$$\begin{aligned} \mathcal{M}_1^{a_0 a_1 a_2} &= p_2^{a_0} p_3^{a_1} \left(2(p_1 \cdot N \cdot \varepsilon_3^A)^{a_2} \mathcal{I}_1 - (p_2 \cdot N \cdot \varepsilon_3^A)^{a_2} \mathcal{I}_2 + (p_2 \cdot V \cdot \varepsilon_3^A)^{a_2} \mathcal{I}_3 - 4(p_3 \cdot V \cdot \varepsilon_3^A)^{a_2} \mathcal{I}_4 \right) \\ &+ p_2^{a_0} p_3 \cdot V \cdot p_3 (\varepsilon_3^A)^{a_1 a_2} \mathcal{I}_4, \end{aligned}$$

$$\begin{aligned} \mathcal{M}_2^{a_0 a_1 a_2} &= p_2^{a_0} p_3^{a_1} \left((p_3 \cdot V \cdot \varepsilon_2^A)^{a_2} \mathcal{I}_3 - (p_3 \cdot N \cdot \varepsilon_2^A)^{a_2} \mathcal{I}_2 \right) \\ &+ \frac{1}{2} p_3^{a_0} (\varepsilon_2^A)^{a_1 a_2} \left(p_2 \cdot V \cdot p_3 \mathcal{I}_3 - p_2 \cdot N \cdot p_3 \mathcal{I}_2 \right), \end{aligned}$$

$$\mathcal{N}_{11}^{a_0 a_1} = p_2^{a_0} \left(2(p_1 \cdot N \cdot \varepsilon_3^S)^{a_1} \mathcal{I}_1 - (p_2 \cdot N \cdot \varepsilon_3^S)^{a_1} \mathcal{I}_2 + (p_2 \cdot V \cdot \varepsilon_3^S)^{a_0} \mathcal{I}_3 - p_3^{a_1} Tr(\varepsilon_3^S \cdot D) \mathcal{I}_4 \right)$$

$$\begin{aligned}
\mathcal{M}_3^{a_0 a_1 a_2} &= 2p_2^{a_0} (\varepsilon_2^A)^{a_1 a_2} \mathcal{I}_1, & \mathcal{M}_4^{a_0 a_1 a_2} &= -p_2^{a_0} (\varepsilon_2^A)^{a_1 a_2} \mathcal{I}_2, \\
\mathcal{M}_5^{a_0 a_1 a_2} &= p_2^{a_0} (\varepsilon_2^A)^{a_1 a_2} \mathcal{I}_3, & \mathcal{M}_6^{a_0 a_1 a_2} &= -p_2^{a_0} (\varepsilon_3^A)^{a_1 a_2} \mathcal{I}_2, \\
\mathcal{M}_7^{a_0 a_1 a_2} &= p_2^{a_0} (\varepsilon_3^A)^{a_1 a_2} \mathcal{I}_3, & \mathcal{N}_1^{a_0 a_1} &= -4p_2^{a_0} p_3^{a_1} \mathcal{I}_1, \\
\mathcal{M}_8^{a_0 a_1} &= -\mathcal{N}_2^{a_0 a_1} = \mathcal{N}_5^{a_0 a_1} = \mathcal{N}_9^{a_0 a_1} = -2p_2^{a_0} p_3^{a_1} \mathcal{I}_2 \\
\mathcal{M}_9^{a_0 a_1} &= \mathcal{N}_3^{a_0 a_1} = -\mathcal{N}_4^{a_0 a_1} = \mathcal{N}_8^{a_0 a_1} = 2p_2^{a_0} p_3^{a_1} \mathcal{I}_3 \\
\mathcal{N}_6^{a_0 a_1} &= 8p_2^{a_0} p_3^{a_1} \mathcal{I}_4, & \mathcal{N}_{10}^{a_0 a_1} &= -2p_2^{a_0} (p_3 \cdot N \cdot \varepsilon_3^S)^{a_1} \mathcal{I}_3 \\
\mathcal{N}_7^{a_0} &= p_2^{a_0} (p_2 \cdot V \cdot p_3 \mathcal{I}_3 - p_2 \cdot N \cdot p_3 \mathcal{I}_2 + 2p_3 \cdot V \cdot p_3 \mathcal{I}_4).
\end{aligned} \tag{18}$$

In above sentences $\mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4$ and \mathcal{I}_7 are the integrals that represent the open and closed string channels as \mathcal{I}_1 in previous section. The explicit form of these integrals is given in [17, 18]. We will study these integrals and their low energy expansion in the last section. They satisfy the relation

$$-2p_1 \cdot N \cdot p_3 \mathcal{I}_1 + 2p_3 \cdot V \cdot p_3 \mathcal{I}_4 + p_2 \cdot N \cdot p_3 \mathcal{I}_2 - p_2 \cdot V \cdot p_3 \mathcal{I}_3 = 0 \tag{19}$$

and similar relation under the interchange of $(2 \leftrightarrow 3)$. The symmetries of the integrals under the interchange of $(2 \leftrightarrow 3)$ are such that \mathcal{I}_1 is invariant, $\mathcal{I}_2 \leftrightarrow \mathcal{I}_3$, and $\mathcal{I}_4 \leftrightarrow \mathcal{I}_7$ [18, 21].

Using the above relations, one finds that the amplitudes (15) and (17) satisfy the Ward identity corresponding to the antisymmetric and the symmetric $NSNS$ gauge transformations respectively. Considering the relation (19), it could be found that the combination of the S-matrix elements involving the last line of $\mathcal{M}_1, \mathcal{M}_2$ and the last term of amplitude (13) satisfies the RR gauge transformations. Also the combination of the S-matrix elements involving the \mathcal{N}_7 and the first term in amplitude (13) satisfies the RR gauge transformations. One can check that the S-matrix elements involving all other terms of $\mathcal{M}_1, \mathcal{M}_2$ and all terms of $\mathcal{M}_8, \mathcal{M}_9, \mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4, \mathcal{N}_5, \mathcal{N}_6, \mathcal{N}_8, \mathcal{N}_9$ are proportional to the RR momentum. This part of amplitude that dose not satisfy the Ward identity of the $NSNS$ gauge transformations, is in terms of RR factor $(f_1)^{a_0 i j k} = p_1^{a_0} \varepsilon_1^{i j k}$. So, it could be found that the amplitudes (15) and (17) would require the amplitude of the RR $(p+1)$ -form with lower transverse indices (in which we will study amplitude of the RR $(p+1)$ -form with two transverse indices in the next section,) to become invariant under the RR gauge transformations.

Regardless the overall factor, these antisymmetric and symmetric amplitudes have been found by T-dual ward identity[21]. T-duality relates these amplitudes to the higher RR form when the Killing coordinate carried by the NSNS polarizations[15, 21].

2.3 $n = 2$ case

From the relation $n = p + 1$ for this RR potential ε_1^{ij} , one gets $p = 1$. The amplitude corresponding to ε_1^{ij} is nonzero only for symmetric NSNS fields and antisymmetric NSNS fields that is inherited from two previous cases. The correlators in $b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b'_{13}, b'_{14}, b'_{15}, b'_{16}$ have non-zero contribution to the amplitude (6). The trace (8) is non-zero only for $m = 4$ and becomes

$$T(2, 1, 4) = 32 \frac{4!}{2!2!} \varepsilon_1^{ij} \epsilon^{a_0 a_1} A_{[i j a_0 a_1]} \quad (20)$$

calculating the sub amplitudes $\mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6, \mathcal{A}_7, \mathcal{A}_8, \mathcal{A}_9, \mathcal{A}_{10}, \mathcal{A}'_{13}, \mathcal{A}'_{14}, \mathcal{A}'_{15}, \mathcal{A}'_{16}$ in (6) and combining them, one finds that the amplitude corresponding to ε_1^{ij} constructed in term of following RR structures:

$$\begin{aligned} & Tr(\varepsilon^A \cdot N \cdot \varepsilon_1 \cdot N), \quad Tr(\varepsilon^A \cdot \varepsilon^A \cdot N \cdot \varepsilon_1 \cdot N), \quad (p \cdot \varepsilon^A \cdot N \cdot \mathcal{E}), \quad \mathcal{E}, \\ & (p \cdot \varepsilon^A \cdot N \cdot \varepsilon_1 \cdot N \cdot \varepsilon^A \cdot p), \quad (\mathcal{E} \cdot N \cdot \varepsilon^A \cdot \varepsilon^A)^\mu, \quad (\mathcal{E} \cdot N \cdot \varepsilon^S)^\mu, \\ & (\varepsilon^S \cdot N \cdot \varepsilon_1 \cdot N \cdot \varepsilon^S)^{\mu\nu}, \quad (\mathcal{E} \cdot N \cdot \varepsilon^S \cdot \varepsilon^S)^\mu, \quad (p \cdot \varepsilon^S \cdot N \cdot \mathcal{E}) \\ & (p \cdot \varepsilon^S \cdot \varepsilon^S \cdot N \cdot \mathcal{E}), \quad (p \cdot \varepsilon^S \cdot N \cdot \varepsilon_1 \cdot N \cdot \varepsilon^S \cdot p), \quad Tr(\varepsilon^S \cdot \varepsilon^S \cdot N \cdot \varepsilon_1 \cdot N), \end{aligned}$$

where μ, ν , as in the previous section, are the world volume indices that contract with the volume $(p+1)$ -form ϵ . In above also, we define $\mathcal{E}^n = (p_i \cdot N \cdot \varepsilon_1)^n$, $i = 2, 3$ that appears in contracting with NSNS polarization in the transverse direction and $\mathcal{E} = p_i \cdot N \cdot \varepsilon_1 \cdot N \cdot p_j$, $i, j = 2, 3, i \neq j$ that only appears in the last structure in the first line.

As mentioned above, the amplitude corresponding to $C^{(p+1)}$ is nonzero when both NSNS polarization tensors are either antisymmetric or symmetric. In the former case, the amplitude corresponding to $(C^{(p+1)})^{ij}$ becomes:

$$\begin{aligned} \mathcal{A}'_A \sim & \epsilon_{a_0 a_1} \left[Tr(\varepsilon_3^A \cdot N \cdot \varepsilon_1 \cdot N) \mathcal{M}'_1{}^{a_0 a_1} + \mathcal{G} Tr((\varepsilon_2^A V \varepsilon_3^A) \cdot N \cdot \varepsilon_1 \cdot N) \mathcal{M}'_2{}^{a_0 a_1} \right. \\ & + (p_1 \cdot N \cdot \varepsilon_3^A \cdot N \cdot \mathcal{E}) \mathcal{M}'_3{}^{a_0 a_1} + (p_1 \cdot N \cdot \varepsilon_2^A \cdot N \cdot \mathcal{E}) \mathcal{M}'_4{}^{a_0 a_1} \\ & + (p_2 \cdot N \cdot \varepsilon_3^A \cdot N \cdot \mathcal{E}) \mathcal{M}'_5{}^{a_0 a_1} + (p_3 \cdot N \cdot \varepsilon_2^A \cdot N \cdot \mathcal{E}) \mathcal{M}'_6{}^{a_0 a_1} \\ & + (p_2 \cdot V \cdot \varepsilon_3^A \cdot N \cdot \mathcal{E}) \mathcal{M}'_7{}^{a_0 a_1} + (p_3 \cdot V \cdot \varepsilon_2^A \cdot N \cdot \mathcal{E}) \mathcal{M}'_8{}^{a_0 a_1} \\ & + (p_3 \cdot N \cdot \varepsilon_2^A \cdot N \cdot \varepsilon_1 \cdot N \cdot \varepsilon_3^A \cdot N \cdot p_2) \mathcal{M}'_9{}^{a_0 a_1} + \mathcal{E} \mathcal{M}'_{10}{}^{a_0 a_1} \\ & + (p_3 \cdot V \cdot \varepsilon_2^A \cdot N \cdot \varepsilon_1 \cdot N \cdot \varepsilon_3^A \cdot V \cdot p_2) \mathcal{M}'_{11}{}^{a_0 a_1} \\ & \left. + \mathcal{G} (\mathcal{E} \cdot N \cdot \varepsilon_2^A \cdot V \cdot \varepsilon_3^A)^{a_1} \mathcal{M}'_{12}{}^{a_0} \right] + (2 \leftrightarrow 3). \quad (21) \end{aligned}$$

In the latter case the amplitude in terms of the symmetric polarization tensors is

$$\begin{aligned}
\mathcal{A}'_S \sim \epsilon_{a_0 a_1} & \left[(\mathcal{E} \cdot N \cdot \varepsilon_3^S)^{a_1} \mathcal{N}'_1^{a_0} + (\varepsilon_2^S \cdot N \cdot \varepsilon_1 \cdot N \cdot \varepsilon_3^S)^{a_0 a_1} \mathcal{N}'_2 + \mathcal{G}(\mathcal{E} \cdot N \cdot \varepsilon_2^S \cdot V \cdot \varepsilon_3^S)^{a_1} \mathcal{N}'_3^{a_0} \right. \\
& + (p_1 \cdot N \cdot \varepsilon_3^S \cdot N \cdot \mathcal{E}) \mathcal{N}'_4^{a_0 a_1} + (p_2 \cdot N \cdot \varepsilon_3^S \cdot N \cdot \mathcal{E}) \mathcal{N}'_5^{a_0 a_1} + (p_3 \cdot N \cdot \varepsilon_2^S \cdot N \cdot \mathcal{E}) \mathcal{N}'_6^{a_0 a_1} \\
& + (p_2 \cdot V \cdot \varepsilon_3^S \cdot N \cdot \mathcal{E}) \mathcal{N}'_7^{a_0 a_1} + (p_3 \cdot V \cdot \varepsilon_2^S \cdot N \cdot \mathcal{E}) \mathcal{N}'_8^{a_0 a_1} + (p_3 \cdot V \cdot \varepsilon_3^S \cdot N \cdot \mathcal{E}) \mathcal{N}'_9^{a_0 a_1} \\
& + \mathcal{G}(p_1 \cdot N \cdot \varepsilon_3^S \cdot V \cdot \varepsilon_2^S \cdot N \cdot \mathcal{E}) \mathcal{N}'_{10}^{a_0 a_1} + \mathcal{G}(p_2 \cdot N \cdot \varepsilon_3^S \cdot V \cdot \varepsilon_2^S \cdot N \cdot \mathcal{E}) \mathcal{N}'_{11}^{a_0 a_1} \\
& + \mathcal{G}(p_2 \cdot V \cdot \varepsilon_3^S \cdot V \cdot \varepsilon_2^S \cdot N \cdot \mathcal{E}) \mathcal{N}'_{12}^{a_0 a_1} + \mathcal{G}(p_3 \cdot V \cdot \varepsilon_3^S \cdot V \cdot \varepsilon_2^S \cdot N \cdot \mathcal{E}) \mathcal{N}'_{13}^{a_0 a_1} \\
& + (p_1 \cdot N \cdot \varepsilon_2^S \cdot N \cdot \varepsilon_1 \cdot N \cdot \varepsilon_3^S)^{a_1} \mathcal{N}'_{14}^{a_0} + (p_2 \cdot V \cdot \varepsilon_2^S \cdot N \cdot \varepsilon_1 \cdot N \cdot \varepsilon_3^S)^{a_1} \mathcal{N}'_{15}^{a_0} \\
& + (p_3 \cdot N \cdot \varepsilon_2^S \cdot N \cdot \varepsilon_1 \cdot N \cdot \varepsilon_3^S)^{a_1} \mathcal{N}'_{16}^{a_0} + (p_3 \cdot V \cdot \varepsilon_2^S \cdot N \cdot \varepsilon_1 \cdot N \cdot \varepsilon_3^S)^{a_1} \mathcal{N}'_{17}^{a_0} \\
& + (p_1 \cdot N \cdot \varepsilon_2^S \cdot N \cdot \varepsilon_1 \cdot N \cdot \varepsilon_3^S \cdot N \cdot p_1) \mathcal{N}'_{18}^{a_0 a_1} + (p_1 \cdot N \cdot \varepsilon_2^S \cdot N \cdot \varepsilon_1 \cdot N \cdot \varepsilon_3^S \cdot N \cdot p_2) \mathcal{N}'_{19}^{a_0 a_1} \\
& + (p_1 \cdot N \cdot \varepsilon_2^S \cdot N \cdot \varepsilon_1 \cdot N \cdot \varepsilon_3^S \cdot V \cdot p_2) \mathcal{N}'_{20}^{a_0 a_1} + (p_1 \cdot N \cdot \varepsilon_2^S \cdot N \cdot \varepsilon_1 \cdot N \cdot \varepsilon_3^S \cdot V \cdot p_3) \mathcal{N}'_{21}^{a_0 a_1} \\
& + (p_2 \cdot V \cdot \varepsilon_2^S \cdot N \cdot \varepsilon_1 \cdot N \cdot \varepsilon_3^S \cdot N \cdot p_2) \mathcal{N}'_{22}^{a_0 a_1} + (p_2 \cdot V \cdot \varepsilon_2^S \cdot N \cdot \varepsilon_1 \cdot N \cdot \varepsilon_3^S \cdot V \cdot p_2) \mathcal{N}'_{23}^{a_0 a_1} \\
& + (p_2 \cdot V \cdot \varepsilon_2^S \cdot N \cdot \varepsilon_1 \cdot N \cdot \varepsilon_3^S \cdot V \cdot p_3) \mathcal{N}'_{24}^{a_0 a_1} + (p_3 \cdot V \cdot \varepsilon_2^S \cdot N \cdot \varepsilon_1 \cdot N \cdot \varepsilon_3^S \cdot N \cdot p_2) \mathcal{N}'_{25}^{a_0 a_1} \\
& \left. + \mathcal{G}Tr((\varepsilon_2^S V \varepsilon_3^S) \cdot N \cdot \varepsilon_1 \cdot N) \mathcal{N}'_{26}^{a_0 a_1} + \mathcal{E} \mathcal{N}'_{27}^{a_0 a_1} \right] + (2 \leftrightarrow 3) \quad (22)
\end{aligned}$$

where the operator \mathcal{G} , which appears in the above amplitudes is defined as (see [21])

$$\begin{aligned}
\mathcal{G}Tr((\varepsilon_n^A \cdot V \cdot \varepsilon_m^A) \cdot N \cdot \varepsilon_1 \cdot N) & \rightarrow Tr((\varepsilon_n^A \cdot V \cdot \varepsilon_m^A) \cdot N \cdot \varepsilon_1 \cdot N) - Tr((\varepsilon_n^A \cdot N \cdot \varepsilon_m^A) \cdot N \cdot \varepsilon_1 \cdot N) \\
\mathcal{G}Tr((\varepsilon_n^S \cdot V \cdot \varepsilon_m^S) \cdot N \cdot \varepsilon_1 \cdot N) & \rightarrow Tr((\varepsilon_n^S \cdot V \cdot \varepsilon_m^S) \cdot N \cdot \varepsilon_1 \cdot N) - Tr((\varepsilon_n^S \cdot N \cdot \varepsilon_m^S) \cdot N \cdot \varepsilon_1 \cdot N) \\
\mathcal{G}(\mathcal{E} \cdot N \cdot \varepsilon_n^A \cdot V \cdot \varepsilon_m^A)^\mu & \rightarrow (\mathcal{E} \cdot N \cdot \varepsilon_n^A \cdot V \cdot \varepsilon_m^A)^\mu - (\mathcal{E} \cdot N \cdot \varepsilon_n^A \cdot N \cdot \varepsilon_m^A)^\mu \\
\mathcal{G}(\mathcal{E} \cdot N \cdot \varepsilon_n^S \cdot V \cdot \varepsilon_m^S)^\mu & \rightarrow (\mathcal{E} \cdot N \cdot \varepsilon_n^S \cdot V \cdot \varepsilon_m^S)^\mu - (\mathcal{E} \cdot N \cdot \varepsilon_n^S \cdot N \cdot \varepsilon_m^S)^\mu \\
\mathcal{G}(p \cdot \varepsilon_n^S \cdot V \cdot \varepsilon_m^S \cdot N \cdot \mathcal{E}) & \rightarrow (p \cdot \varepsilon_n^S \cdot V \cdot \varepsilon_m^S \cdot N \cdot \mathcal{E}) - (p \cdot \varepsilon_n^S \cdot N \cdot \varepsilon_m^S \cdot N \cdot \mathcal{E})
\end{aligned}$$

where n, m are the particle labels of NSNS polarization tensors. Acting this operator on the above structures, make them T-dual invariant. It should also be pointed out that the T-dual effect of this operator only works for the structures appear in the third and forth line in which the Killing coordinate dose not carry with the index μ . The elements of the amplitude with no RR contribution make the terms $\mathcal{M}_i ; i = 1, \dots, 12$ appear in the antisymmetric amplitude (21) and $\mathcal{N}_i ; i = 1, \dots, 27$ appear in the symmetric amplitude (22). These terms are the following:

$$\begin{aligned}
\mathcal{M}'_1^{a_0 a_1} & = -\frac{1}{4}(p_3 \cdot V \cdot \varepsilon_2^A)^{a_1} \left(p_1 \cdot N \cdot p_2 p_2^{a_0} \mathcal{I}_3 + p_1 \cdot N \cdot p_3 p_3^{a_0} \mathcal{I}_2 \right) \\
& - \frac{1}{4}(p_3 \cdot N \cdot \varepsilon_2^A)^{a_1} \left(p_1 \cdot N \cdot p_3 p_2^{a_0} \mathcal{I}_3 + p_1 \cdot N \cdot p_2 p_3^{a_0} \mathcal{I}_2 \right) \\
& - p_1 \cdot N \cdot p_3 p_3^{a_0} \left((p_2 \cdot V \cdot \varepsilon_2^A)^{a_1} \mathcal{I}_7 + \frac{1}{2}(p_1 \cdot N \cdot \varepsilon_2^A)^{a_1} \mathcal{I}_1 \right) \\
& - \frac{1}{8}(\varepsilon_2^A)^{a_0 a_1} \left(2p_1 \cdot N \cdot p_3 p_2 \cdot V \cdot p_2 \mathcal{I}_7 + p_1 \cdot N \cdot p_2 p_2 \cdot V \cdot p_3 \mathcal{I}_3 - p_1 \cdot N \cdot p_2 p_2 \cdot N \cdot p_3 \mathcal{I}_2 \right) \\
& + \frac{1}{2}p_2^{a_0} p_3^{a_1} \left(p_3 \cdot V \cdot \varepsilon_2^A \cdot V \cdot p_2 \mathcal{J}_1 - \frac{1}{2}p_3 \cdot V \cdot \varepsilon_2^A \cdot N \cdot p_1 \mathcal{I}_3 + p_2 \cdot V \cdot \varepsilon_2^A \cdot N \cdot p_3 \mathcal{J}_2 \right)
\end{aligned}$$

$$\begin{aligned}
& -p_3 \cdot V \cdot \varepsilon_2^A \cdot N \cdot p_3 (\mathcal{J} + \mathcal{J}_5) + \frac{1}{2} p_3 \cdot N \cdot \varepsilon_2^A \cdot N \cdot p_1 \mathcal{I}_2 \Big) \\
& - \frac{1}{4} p_1^{a_0} (p_3 \cdot V \cdot \varepsilon_2^A)^{a_1} \Big(p_2 \cdot V \cdot p_2 \mathcal{J}_1 + p_3 \cdot V \cdot p_3 \mathcal{J}_4 - 2 p_2 \cdot N \cdot p_3 \mathcal{J} \Big) \\
& - \frac{1}{2} p_1^{a_0} p_3 \cdot V \cdot p_3 \Big((p_1 \cdot N \cdot \varepsilon_2^A)^{a_1} \mathcal{I}_4 - (p_2 \cdot V \cdot \varepsilon_2^A)^{a_1} \mathcal{J}_3 \Big) \\
& - \frac{1}{4} p_1^{a_0} (p_3 \cdot N \cdot \varepsilon_2^A)^{a_1} \Big(\mathcal{J}_{15} p_2 \cdot N \cdot p_3 + (\mathcal{J}_{16} - 2 \mathcal{J}) p_2 \cdot V \cdot p_3 \Big), \\
\mathcal{M}'_2^{a_0 a_1} &= \frac{1}{2} p_2^{a_0} p_3^{a_1} p_2 \cdot V \cdot p_3 \mathcal{J}, \\
\mathcal{M}'_3^{a_0 a_1} &= \frac{1}{4} \Big(p_2^{a_0} (p_3 \cdot V \cdot \varepsilon_2^A)^{a_1} \mathcal{I}_3 - 4 p_2^{a_0} (p_2 \cdot V \cdot \varepsilon_2^A)^{a_1} \mathcal{I}_4 + 2 p_2^{a_0} (p_1 \cdot N \cdot \varepsilon_2^A)^{a_1} \mathcal{I}_1 \\
& + p_3^{a_0} (p_3 \cdot N \cdot \varepsilon_2^A)^{a_1} \mathcal{I}_2 + \frac{1}{2} (\varepsilon_2^A)^{a_0 a_1} (p_2 \cdot V \cdot p_3 \mathcal{I}_3 - p_2 \cdot N \cdot p_3 \mathcal{I}_2) \Big), \\
\mathcal{M}'_4^{a_0 a_1} &= \frac{1}{8} (\varepsilon_3^A)^{a_0 a_1} \Big(2 p_2 \cdot V \cdot p_3 \mathcal{I}_4 + p_2 \cdot V \cdot p_3 \mathcal{I}_3 - p_2 \cdot N \cdot p_3 \mathcal{I}_2 - 2 p_3 \cdot V \cdot p_3 \mathcal{I}_4 - 4 p_1 \cdot N \cdot p_3 \mathcal{I}_2 \Big) \\
& - \frac{1}{4} \Big(p_2^{a_0} (p_2 \cdot V \cdot \varepsilon_3^A)^{a_1} \mathcal{I}_3 + p_3^{a_0} (p_2 \cdot N \cdot \varepsilon_3^A)^{a_1} \mathcal{I}_2 - 4 p_2^{a_0} (p_3 \cdot V \cdot \varepsilon_3^A)^{a_1} \mathcal{I}_4 \\
& - 2 p_2^{a_0} (p_1 \cdot N \cdot \varepsilon_3^A)^{a_1} \mathcal{I}_1 \Big), \\
\mathcal{M}'_5^{a_0 a_1} &= -\frac{1}{4} p_1^{a_0} \Big((p_3 \cdot N \cdot \varepsilon_2^A)^{a_1} (\mathcal{J}_{16} - 2 \mathcal{J}_5) + (p_3 \cdot V \cdot \varepsilon_2^A)^{a_1} \mathcal{J}_{15} \Big), \\
\mathcal{M}'_6^{a_0 a_1} &= \frac{1}{4} p_1 \cdot N \cdot p_3 (\varepsilon_3^A)^{a_0 a_1} \mathcal{I}_3, \\
\mathcal{M}'_7^{a_0 a_1} &= -\frac{1}{4} p_1^{a_0} \Big((p_3 \cdot N \cdot \varepsilon_2^A)^{a_1} \mathcal{J}_{15} + (p_3 \cdot V \cdot \varepsilon_2^A)^{a_1} (\mathcal{J}_{16} - 4 \mathcal{J} + 2 \mathcal{J}_5) \Big), \\
\mathcal{M}'_8^{a_0 a_1} &= \frac{1}{4} p_1 \cdot N \cdot p_3 (\varepsilon_3^A)^{a_0 a_1} \mathcal{I}_2, \\
\mathcal{M}'_9^{a_0 a_1} &= -\mathcal{M}'_{11}^{a_0 a_1} = -\frac{1}{2} p_2^{a_0} p_3^{a_1} \mathcal{J}, \\
\mathcal{M}'_{10}^{a_0 a_1} &= \frac{1}{4} (\varepsilon_2^A)^{a_0 a_1} \Big((p_1 \cdot N \cdot \varepsilon_3^A \cdot N \cdot p_2) \mathcal{I}_2 - (p_1 \cdot N \cdot \varepsilon_3^A \cdot V \cdot p_2) - 2 (p_1 \cdot N \cdot \varepsilon_3^A \cdot N \cdot p_1) \mathcal{I}_1 \Big) \\
& - \frac{1}{4} p_1^{a_0} \Big(\mathcal{G}(p_2 \cdot N \cdot \varepsilon_3^A \cdot V \cdot \varepsilon_2^A)^{a_1} \mathcal{J}_{15} + \mathcal{G}(p_2 \cdot V \cdot \varepsilon_3^A \cdot N \cdot \varepsilon_2^A)^{a_1} \mathcal{J}_{15} \\
& - \mathcal{G}(p_2 \cdot V \cdot \varepsilon_3^A \cdot V \cdot \varepsilon_2^A)^{a_1} (4 \mathcal{J} - \mathcal{J}_{16} - 2 \mathcal{J}_5) + \mathcal{G}(p_2 \cdot N \cdot \varepsilon_3^A \cdot N \cdot \varepsilon_2^A)^{a_1} (\mathcal{J}_{16} - 2 \mathcal{J}_5) \Big)
\end{aligned}$$

$$\begin{aligned}
& -2\left(p_1^{a_0}\mathcal{G}(p_1\cdot N\cdot\varepsilon_3^A\cdot N\cdot\varepsilon_2^A)^{a_1}\mathcal{I}_2 + p_2^{a_0}\mathcal{G}(p_1\cdot N\cdot\varepsilon_3^A\cdot V\cdot\varepsilon_2^A)^{a_1}\mathcal{I}_3\right) \\
& -\frac{1}{2}p_2^{a_0}p_3^{a_1}\left((2\mathcal{J}-\mathcal{J}_{16})(\text{Tr}(\varepsilon_2^A\cdot N\cdot\varepsilon_3^A\cdot N) + \text{Tr}(\varepsilon_2^A\cdot V\cdot\varepsilon_3^A\cdot V))\right. \\
& \left.-2\mathcal{J}_{15}\text{Tr}(\varepsilon_2^A\cdot V\cdot\varepsilon_3^A\cdot N)\right),
\end{aligned}$$

$$\mathcal{M}'_{12}{}^{a_0} = \frac{1}{2}\left(p_2^{a_0}p_1\cdot N\cdot p_3\mathcal{I}_2 + p_1^{a_0}p_3\cdot V\cdot p_3\mathcal{J}_{12}\right),$$

$$\begin{aligned}
\mathcal{N}'_1{}^{a_0} &= \frac{1}{2}p_2^{a_0}\left(p_1\cdot N\cdot\varepsilon_2^S\cdot V\cdot p_3\mathcal{I}_3 + 4p_1\cdot N\cdot\varepsilon_2^S\cdot V\cdot p_2\mathcal{I}_4 - 2p_1\cdot N\cdot\varepsilon_2^S\cdot N\cdot p_1\mathcal{I}_1\right) \\
& -p_3^{a_0}p_1\cdot N\cdot\varepsilon_2^S\cdot N\cdot p_3\mathcal{I}_2 - \frac{1}{4}(p_1\cdot N\cdot\varepsilon_2^S)^{a_0}\left(2p_3\cdot V\cdot p_3\mathcal{I}_4 + p_2\cdot V\cdot p_3\mathcal{I}_3 - p_2\cdot N\cdot p_3\mathcal{I}_2\right) \\
& -\frac{1}{2}(p_3\cdot N\cdot\varepsilon_2^S)^{a_0}p_1\cdot N\cdot p_3\mathcal{I}_3 + \frac{1}{2}(p_3\cdot V\cdot\varepsilon_2^S)^{a_0}p_1\cdot N\cdot p_3\mathcal{I}_2 + p_2^{a_0}\text{Tr}(\varepsilon_2^A\cdot V)p_1\cdot N\cdot p_3\mathcal{I}_7 \\
& -\frac{1}{4}p_1^{a_0}\left(2p_2\cdot V\cdot\varepsilon_3^S\cdot N\cdot p_2\mathcal{J}_{15} - 2(2\mathcal{I}_4p_1\cdot N\cdot p_3 - \mathcal{J}_4p_2\cdot N\cdot p_3 + \mathcal{J}_{12}p_2\cdot V\cdot p_3)\text{Tr}[\varepsilon_3^S\cdot V]\right. \\
& + p_2\cdot V\cdot\varepsilon_3^S\cdot V\cdot p_2(\mathcal{J}_{16} - 4\mathcal{J} + 2\mathcal{J}_5) + 4p_3\cdot V\cdot\varepsilon_3^S\cdot V\cdot p_2\mathcal{J}_{12} - 2p_2\cdot V\cdot\varepsilon_3^S\cdot N\cdot p_1\mathcal{I}_3 \\
& \left.+ p_2\cdot N\cdot\varepsilon_3^S\cdot N\cdot p_2(\mathcal{J}_{16} - 2\mathcal{J}_5) - 4\mathcal{J}_4p_3\cdot V\cdot\varepsilon_3^S\cdot N\cdot p_2 + 4\mathcal{I}_2p_2\cdot N\cdot\varepsilon_3^S\cdot N\cdot p_1\right),
\end{aligned}$$

$$\mathcal{N}'_2 = -\frac{1}{4}\left(2p_1\cdot N\cdot p_2p_3\cdot V\cdot p_3\mathcal{I}_4 + p_1\cdot N\cdot p_3p_2\cdot V\cdot p_3\mathcal{I}_2 - p_1\cdot N\cdot p_3p_2\cdot N\cdot p_3\mathcal{I}_3\right),$$

$$\mathcal{N}'_3{}^{a_0} = \frac{1}{2}\left(p_1^{a_0}p_3\cdot V\cdot p_3\mathcal{J}_{12} + p_2^{a_0}p_1\cdot N\cdot p_3\mathcal{I}_2\right),$$

$$\mathcal{N}'_4{}^{a_0a_1} = p_2^{a_0}(p_1\cdot N\cdot\varepsilon_2^S)^{a_1}\mathcal{I}_1,$$

$$\mathcal{N}'_5{}^{a_0a_1} = -\frac{1}{2}p_3^{a_0}(p_1\cdot N\cdot\varepsilon_2^S)^{a_1}\mathcal{I}_2,$$

$$\begin{aligned}
\mathcal{N}'_6{}^{a_0a_1} &= -\frac{1}{4}p_1^{a_0}\left(2p_3^{a_1}\text{Tr}[\varepsilon_3^S\cdot V]\mathcal{J}_4 - 2(p_1\cdot N\cdot\varepsilon_3^S)^{a_1}\mathcal{I}_2 - (p_2\cdot N\cdot\varepsilon_3^S)^{a_1}(\mathcal{J}_{16} - 2\mathcal{J}_5)\right. \\
& \left.- (p_2\cdot V\cdot\varepsilon_3^S)^{a_1}\mathcal{J}_{15}\right),
\end{aligned}$$

$$\mathcal{N}'_7{}^{a_0a_1} = -\frac{1}{2}p_2^{a_0}(p_1\cdot N\cdot\varepsilon_2^S)^{a_1}\mathcal{I}_3,$$

$$\begin{aligned}
\mathcal{N}'_8{}^{a_0a_1} &= \frac{1}{4}p_1^{a_0}\left(2p_3^{a_1}\text{Tr}[\varepsilon_3^S\cdot V]\mathcal{J}_{12} + (p_2\cdot N\cdot\varepsilon_3^S)^{a_1}\mathcal{J}_{15} + (p_2\cdot V\cdot\varepsilon_3^S)^{a_1}(\mathcal{J}_{16} - 4\mathcal{J} + 2\mathcal{J}_5)\right) \\
& -\frac{1}{2}p_2^{a_0}(p_1\cdot N\cdot\varepsilon_3^S)^{a_1}\mathcal{I}_3,
\end{aligned}$$

$$\mathcal{N}'^{a_0 a_1}_9 = 2p_2^{a_0} (p_1 \cdot N \cdot \varepsilon_2^S)^{a_1} \mathcal{I}_4,$$

$$\mathcal{N}'^{a_0 a_1}_{10} = -\frac{1}{2} p_2^{a_0} p_3^{a_1} (\mathcal{I}_3 + \mathcal{I}_2),$$

$$\mathcal{N}'^{a_0 a_1}_{11} = \frac{1}{4} p_2^{a_0} p_3^{a_1} (\mathcal{J}_{15} - \mathcal{J}_{16} + 2\mathcal{J}_5),$$

$$\mathcal{N}'^{a_0 a_1}_{12} = \frac{1}{4} p_2^{a_0} p_3^{a_1} (\mathcal{J}_{16} - 4\mathcal{J} + 2\mathcal{J}_5 - \mathcal{J}_{15}),$$

$$\mathcal{N}'^{a_0 a_1}_{13} = p_2^{a_0} p_3^{a_1} (\mathcal{J}_{12} + \mathcal{J}_4),$$

$$\mathcal{N}'^{a_0}_{14} = p_1^{a_0} p_3 \cdot V \cdot p_3 \mathcal{I}_4 - \frac{1}{2} p_3^{a_0} p_1 \cdot N \cdot p_3 \mathcal{I}_1,$$

$$\mathcal{N}'^{a_0}_{15} = 2 \left(p_1^{a_0} p_3 \cdot V \cdot p_3 \mathcal{J}_3 + p_3^{a_0} p_1 \cdot N \cdot p_3 \mathcal{I}_7 \right),$$

$$\mathcal{N}'^{a_0}_{16} = -\frac{1}{2} \left(p_1^{a_0} (\mathcal{J}_{15} p_2 \cdot N \cdot p_3 + (\mathcal{J}_{16} - 2\mathcal{J}) p_2 \cdot V \cdot p_3) + p_3^{a_0} p_1 \cdot N \cdot p_2 \mathcal{I}_2 + p_2^{a_0} p_1 \cdot N \cdot p_3 \mathcal{I}_3 \right),$$

$$\mathcal{N}'^{a_0}_{17} = \frac{1}{2} \left(p_1^{a_0} (p_2 \cdot V \cdot p_2 \mathcal{J}_1 + p_3 \cdot V \cdot p_3 \mathcal{J}_4 - 2p_2 \cdot N \cdot p_3 \mathcal{J}) + p_1 \cdot N \cdot p_2 p_2^{a_0} \mathcal{I}_3 + p_1 \cdot N \cdot p_3 p_3^{a_0} \mathcal{I}_2 \right),$$

$$\mathcal{N}'^{a_0 a_1}_{18} = \frac{1}{2} p_2^{a_0} p_3^{a_1} \mathcal{I}_1, \quad \mathcal{N}'^{a_0 a_1}_{19} = -\frac{1}{2} p_2^{a_0} p_3^{a_1} \mathcal{I}_2, \quad \mathcal{N}'^{a_0 a_1}_{20} = \frac{1}{2} p_2^{a_0} p_3^{a_1} \mathcal{I}_3,$$

$$\mathcal{N}'^{a_0 a_1}_{21} = -2p_2^{a_0} p_3^{a_1} \mathcal{I}_4, \quad \mathcal{N}'^{a_0 a_1}_{22} = p_2^{a_0} p_3^{a_1} \mathcal{J}_2, \quad \mathcal{N}'^{a_0 a_1}_{23} = -p_2^{a_0} p_3^{a_1} \mathcal{J}_1,$$

$$\mathcal{N}'^{a_0 a_1}_{24} = 2p_2^{a_0} p_3^{a_1} \mathcal{J}_3, \quad \mathcal{N}'^{a_0 a_1}_{25} = p_2^{a_0} p_3^{a_1} \mathcal{J}_5, \quad \mathcal{N}'^{a_0 a_1}_{26} = -\frac{1}{2} p_2^{a_0} p_3^{a_1} p_2 \cdot N \cdot p_3 \mathcal{J},$$

$$\begin{aligned} \mathcal{N}'^{a_0 a_1}_{27} = & -\frac{1}{2} \text{Tr}(\varepsilon_3^S \cdot V) \left(-2p_3^{a_0} \mathcal{I}_4 (p_1 \cdot N \cdot \varepsilon_2^S)^{a_1} \right. \\ & + p_1^{a_0} (\mathcal{J}_4 (p_3 \cdot V \cdot \varepsilon_2^S)^{a_1} - \mathcal{J}_{12} (p_3 \cdot N \cdot \varepsilon_2^S)^{a_1} + \mathcal{J}_3 p_3^{a_1} \text{Tr}(\varepsilon_2^S \cdot V)) \Big) \\ & - \frac{1}{2} (p_1 \cdot N \cdot \varepsilon_2^S)^{a_0} \left((p_2 \cdot N \cdot \varepsilon_3^S)^{a_1} \mathcal{I}_2 + (p_2 \cdot V \cdot \varepsilon_3^S)^{a_1} \mathcal{I}_3 - 2(p_1 \cdot N \cdot \varepsilon_3^S)^{a_1} \mathcal{I}_1 \right) \\ & - \frac{1}{2} p_2^{a_0} p_3^{a_1} \left(-2\text{Tr}(\varepsilon_2^S \cdot V \cdot \varepsilon_3^S \cdot N) (2\mathcal{J} - \mathcal{J}_{16}) + \text{Tr}(\varepsilon_2^S \cdot N \cdot \varepsilon_3^S \cdot N) \mathcal{J}_{15} \right. \\ & \left. + \text{Tr}(\varepsilon_2^S \cdot V \cdot \varepsilon_3^S \cdot V) \mathcal{J}_{15} \right) + 2p_3^{a_0} \left(\mathcal{G} (p_1 \cdot N \cdot \varepsilon_2^S \cdot V \cdot \varepsilon_3^S)^{a_1} \mathcal{I}_3 \right). \end{aligned}$$

In the above sentences \mathcal{J} and \mathcal{J}_i 's are some other integrals as \mathcal{I}_i 's in the previous sections in which the explicit forms are given in [17, 21]. The symmetries of the integrals under

the interchange of $(2 \leftrightarrow 3)$ are such that $\mathcal{J}, \mathcal{J}_3, \mathcal{J}_{15}, \mathcal{J}_{16}$ are invariant, $\mathcal{J}_1 \leftrightarrow \mathcal{J}_4$, $\mathcal{J}_2 \leftrightarrow \mathcal{J}_{12}$, and $\mathcal{J}_5 \leftrightarrow -\mathcal{J}_5$. The antisymmetric amplitude (21) and the symmetric amplitude (22) satisfy the Ward identity corresponding to the B-field and graviton gauge transformations respectively, provided the integrals satisfy the relations

$$\begin{aligned} -2\mathcal{I}_3 p_1 \cdot N \cdot p_3 + \mathcal{J}_{15} p_2 \cdot N \cdot p_3 + 2\mathcal{J}_{12} p_3 \cdot V \cdot p_3 + (-4\mathcal{J} + \mathcal{J}_{16} + 2\mathcal{J}_5) p_2 \cdot V \cdot p_3 &= 0 \\ 2\mathcal{I}_2 p_1 \cdot N \cdot p_3 - 2\mathcal{J}_4 p_3 \cdot V \cdot p_3 + \mathcal{J}_{15} p_2 \cdot V \cdot p_3 + (\mathcal{J}_{16} - 2\mathcal{J}_5) p_2 \cdot N \cdot p_3 &= 0 \\ -2\mathcal{I}_7 p_1 \cdot N \cdot p_3 + \mathcal{J}_2 p_2 \cdot N \cdot p_3 + 2\mathcal{J}_3 p_3 \cdot V \cdot p_3 - \mathcal{J}_1 p_2 \cdot V \cdot p_3 &= 0 \end{aligned} \quad (23)$$

and similar relations under the interchange of $(2 \leftrightarrow 3)$. We check the above identities by performing the integrals explicitly.

The amplitudes (21) and (22) are not invariant under the RR ward identity. Combining the part of the amplitudes (21) and (22) involving the first three lines in \mathcal{N}'_1 , the last terms in $\mathcal{N}'_3, \mathcal{N}'_8, \mathcal{N}'_{14}, \mathcal{N}'_{15}$, the last two terms in $\mathcal{N}'_{16}, \mathcal{N}'_{17}$, the terms in $\mathcal{N}'_4, \mathcal{N}'_5, \mathcal{N}'_7, \mathcal{N}'_9$, the first term in \mathcal{N}'_{27} , the first four lines in \mathcal{M}'_1 , the terms in $\mathcal{M}'_3, \mathcal{M}'_4, \mathcal{M}'_6, \mathcal{M}'_8$ and the first term in \mathcal{M}'_{12} with the amplitude corresponding to $C^{(p+1)}$ with three transverse indices, which has been found in the previous section, satisfy the RR Ward identity. The other terms in all \mathcal{N}'_i 's and \mathcal{M}'_i 's proportional to RR momentum p_1 . The amplitude corresponding to $C^{(p+1)}$ with three transverse indices has also some terms that satisfy the RR Ward identity after combining with the amplitude corresponding to $C^{(p+1)}$ with four transverse indices which has been found in the section (2.1). However the RR invariant amplitude dose not satisfy the Ward identity associated with the NSNS gauge transformations anymore. The asymmetry of the amplitudes corresponding to $C^{(p+1)}$ with four, three and two transverse indices under the RR gauge transformations indicates that one has to include the amplitude corresponding to $C^{(p+1)}$ with one transverse index in which we are not interested.

The polarization tensors in all terms except the terms including the operator \mathcal{G} , are contracted with the momentum or with the world volume form. On the other hand, as we mentioned before, the operator \mathcal{G} generate the T-dual invariant structure. So the amplitudes (21) and (22) invariant under linear T-duality when the Killing coordinate is an index of the RR potential. T-duality relates the amplitudes to the higher RR form when the Killing coordinate carries by the NSNS polarizations.

Applying the linear T-duality on the amplitude of $C^{(p-1)}$ that has one transverse index, in which has been found using T-dual Ward identity in [21] and explicitly in [19], when the Killing coordinate is carried by the NSNS polarization tensors, regardless the overall factor, one can find the amplitude (21) and (22) except the elements constructed with the last four lines in \mathcal{M}'_{10} , the second line, the fourth line and the first term in the last line in \mathcal{N}'_{27} . It could be found that T-duality can only predict the structures contribute to these above-mentioned terms in \mathcal{M}'_{10} and \mathcal{N}'_{27} and can not capture them with the right integral functions.

3 Four derivative couplings

The amplitudes corresponding to RR $(p+1)$ -form potential $C^{(p+1)}$ with four, three and two transverse indices that we have found in this paper are invariant under the Ward identity associated with the B-field and graviton gauge transformations. Hence, it could be rewrite these amplitudes in terms of the field strength of the NSNS states. One can rewrite the amplitude (13) in terms of H and R as

$$\begin{aligned} \mathcal{A} \sim & T_3 \epsilon_{a_0 \dots a_3} \left((\mathcal{R}_2 \cdot N \cdot N \cdot \epsilon_1 \cdot N \cdot N \cdot \mathcal{R}_3)^{a_0 a_1 a_2 a_3} + \frac{1}{6} p_2^{a_0} \text{Tr}(\mathcal{E} \cdot N \cdot N \cdot N \cdot H_2 \cdot N \cdot N \cdot N) H_3^{a_1 a_2 a_3} \right) \\ & \times \mathcal{I}_1 \delta(p_1^a + p_2^a + p_3^a) + (2 \leftrightarrow 3) \end{aligned} \quad (24)$$

where $\mathcal{E}^{nmr} = (p_3 \cdot N \cdot \epsilon_1)^{nmr}$. Our notation for two(three) consecutive projection operators $N \cdot N (N \cdot N \cdot N)$ in the above amplitude and in the following in this paper is that the two (three) transverse legs of involving tensors contracted with each other *i.e.*, $(\epsilon_1)_{ijkl} \mathcal{R}_2^{a_0 a_1 ij} \mathcal{R}_3^{a_2 a_3 kl} = (\mathcal{R}_2 \cdot N \cdot N \cdot \epsilon_1 \cdot N \cdot N \cdot \mathcal{R}_3)^{a_0 a_1 a_2 a_3}$ and $p_3^l (\epsilon_1)_{ijkl} H_2^{ijk} = \text{Tr}(\mathcal{E} \cdot N \cdot N \cdot N \cdot H_2 \cdot N \cdot N \cdot N)$.

The next amplitude that we want to write in terms of NSNS fields strength is the $(\epsilon_1^{(p+1)})^{ijk}$ amplitude. This amplitude has two parts, the antisymmetric amplitude (15) and symmetric amplitude (17). Using the identity (19), one can rewrite the former amplitude in terms of H_2 and H_3 as

$$\begin{aligned} \mathcal{A}^A \sim & T_2 \epsilon_{a_0 a_1 a_2} \left[\left(2p_1 \cdot N \cdot H_3 \cdot N \cdot N \cdot \mathcal{E} \mathcal{I}_1 - p_2 \cdot N \cdot H_3 \cdot N \cdot N \cdot \mathcal{E} \mathcal{I}_2 + p_2 \cdot V \cdot H_3 \cdot N \cdot N \cdot \mathcal{E} \mathcal{I}_3 \right) H_2^{a_2 a_1 a_0} \right. \\ & + p_2^{a_2} \left(2(p_1 \cdot N \cdot H_3)^{a_1 a_0} \mathcal{I}_1 - (p_2 \cdot N \cdot H_3)^{a_1 a_0} \mathcal{I}_2 + (p_2 \cdot V \cdot H_3)^{a_1 a_0} \mathcal{I}_3 - 4(p_3 \cdot V \cdot H_3)^{a_1 a_0} \mathcal{I}_4 \right. \\ & \left. \left. - \frac{1}{3} H_3^{a_2 a_1 a_0} ((p_2 \cdot N \cdot p_3) \mathcal{I}_2 - (p_2 \cdot V \cdot p_3) \mathcal{I}_3) \right) \text{Tr}(\epsilon_1 \cdot N \cdot N \cdot N \cdot H_2 \cdot N \cdot N \cdot N) \right. \\ & \left. + 3p_2^{a_2} \left((\mathcal{E} \cdot N \cdot N \cdot H_2 \cdot V \cdot H_3)^{a_1 a_0} \mathcal{I}_2 - (\mathcal{E} \cdot N \cdot N \cdot H_2 \cdot N \cdot H_3)^{a_1 a_0} \mathcal{I}_3 \right) \right] + (2 \leftrightarrow 3) \end{aligned} \quad (25)$$

where $\mathcal{E}^{nm} = (p_2 \cdot N \cdot \epsilon_1)^{nm}$ in the first line and $\mathcal{E}^{nm} = (p_3 \cdot N \cdot \epsilon_1)^{nm}$ in the last line. As we know, the metric contribute to curvature tensor and covariant derivative in the effective action. The metric in the effective action also appears in contracting the indices. Because of these points, one dose not expect that all the contact terms of the symmetric amplitude (17) to be rewritten as $R_2 R_3$. So we left with this amplitude in terms of it's symmetric NSNS polarizations.

Before going to the next amplitude that we found in the section (2.3), Let us analyze the above amplitudes at low energy to find the coupling of one RR $(p+1)$ -form and two NSNS states. So we need the α' expansion of the integral functions that appear in the amplitudes. These low energy expansion contain massless poles as well as contact terms that have been found in [5].

Some contact terms, that may come from the subtraction of field theory massless poles from the string theory amplitude, add to the contact terms of the string theory amplitude. For antisymmetric amplitudes which one can write in terms of the antisymmetric NSNS fields, it may be expected that this additional contact terms to be avoided when one writes the string theory amplitude, in terms of H [17, 18].

Using this point and considering the contact terms in the low energy expansion of the integrals functions, one can find couplings at order α'^2 . It has been shown in [5] that the low energy expansion of the integral functions \mathcal{I}_1 , \mathcal{I}_2 and \mathcal{I}_4 have no contact term contribution. So the low energy limit of the amplitudes (24) and (25) contain any couplings.

Now consider the last case of the amplitudes in this paper where the RR $p + 1$ -form carries two transverse indices in which we found it as antisymmetric amplitude (21) and symmetric amplitude (22). In the latter amplitude also one cannot rewrite it in terms of curvatures tensors. In the former amplitude, it can be rewritten in terms of antisymmetric NSNS fields strength. The integral functions \mathcal{J}_3 and \mathcal{J}_5 also have any contact term in their low energy expansions [5]. Hence, without considering the terms containing the integral functions \mathcal{J}_3 and \mathcal{J}_5 , one can rewrite the amplitude (21) in terms of H_2 and H_3 . The result is

$$\begin{aligned}
\mathcal{A}^A \sim & T_1 \epsilon_{a_0 a_1} \left[(-4\mathcal{J} + \mathcal{J}_{16}) p_3^{a_0} (H_2 \cdot V \cdot V \cdot H_3 \cdot N \cdot \mathcal{E})^{a_1} + 2\mathcal{J}_{15} p_3^{a_0} (\mathcal{E} \cdot N \cdot H_3 \cdot V \cdot N \cdot H_2)^{a_1} \right. \\
& + (-4\mathcal{J} + \mathcal{J}_{16}) p_3^{a_0} (\mathcal{E} \cdot N \cdot H_3 \cdot N \cdot N \cdot H_2)^{a_1} + 4\mathcal{J}_2 p_3^{a_0} (p_2 \cdot V \cdot H_2 \cdot N \cdot H_3 \cdot N \cdot N \cdot \varepsilon_1)^{a_1} \\
& - 4\mathcal{J}_1 p_3^{a_0} (p_2 \cdot V \cdot H_2 \cdot V \cdot H_3 \cdot N \cdot N \cdot \varepsilon_1)^{a_1} + \mathcal{J}_{15} p_3^{a_0} (p_3 \cdot N \cdot H_2 \cdot N \cdot H_3 \cdot N \cdot N \cdot \varepsilon_1)^{a_1} \\
& - 8\mathcal{J}_2 p_2^{a_0} p_3 \cdot N \cdot H_2 \cdot N \cdot N \cdot H_3^{a_1} \cdot N \cdot \varepsilon_1 \cdot \mathcal{N} + \mathcal{J}_{16} p_2 \cdot N \cdot H_3 \cdot \mathcal{N} \cdot N \cdot H_2^{a_0 a_1} \mathcal{E} \cdot \mathcal{N} \\
& + \mathcal{J}_{16} p_3^{a_0} (\varepsilon_1 \cdot N \cdot N \cdot H_3 \cdot V \cdot H_2 \cdot N \cdot p_3)^{a_1} + \mathcal{J}_{15} p_2 \cdot N \cdot H_3 \cdot \mathcal{N} \cdot V \cdot H_2^{a_0 a_1} \mathcal{E} \cdot \mathcal{N} \\
& + (-4\mathcal{J} + \mathcal{J}_{16}) p_3^{a_0} (\varepsilon_1 \cdot N \cdot N \cdot H_3 \cdot N \cdot H_2 \cdot V \cdot p_3)^{a_1} + \mathcal{J}_{15} p_2 \cdot V \cdot H_3 \cdot \mathcal{N} \cdot N \cdot H_2^{a_0 a_1} \\
& + \mathcal{J}_{15} p_3^{a_0} (\varepsilon_1 \cdot N \cdot N \cdot H_3 \cdot V \cdot H_2 \cdot V \cdot p_3)^{a_1} + 8\mathcal{J}_2 p_2^{a_0} p_3 \cdot V \cdot H_2 \cdot \mathcal{N} \cdot V \cdot H_3^{a_1} \cdot N \cdot \varepsilon_1 \cdot \mathcal{N} \\
& + \mathcal{J}_{12} p_3 \cdot V \cdot p_3 (\varepsilon_1 \cdot N \cdot N \cdot H_2 \cdot N \cdot H_3)^{a_0 a_1} + (-4\mathcal{J} + \mathcal{J}_{16}) p_2 \cdot V \cdot H_3 \cdot \mathcal{N} \cdot V \cdot H_2^{a_0 a_1} \\
& - \mathcal{J}_4 p_3 \cdot V \cdot p_3 (\varepsilon_1 \cdot N \cdot N \cdot H_2 \cdot V \cdot H_3)^{a_0 a_1} - \frac{1}{16} (p_3 \cdot N \cdot H_2 \cdot N \cdot N \cdot \varepsilon_1) \left((p_2 \cdot N \cdot H_3)^{a_0 a_1} \mathcal{J}_{15} \right. \\
& \left. - (p_2 \cdot V \cdot H_3)^{a_0 a_1} (4\mathcal{J} - \mathcal{J}_{16}) + 4(p_3 \cdot V \cdot H_3)^{a_0 a_1} \mathcal{J}_{12} \right) - \frac{1}{16} (p_3 \cdot V \cdot H_2 \cdot N \cdot N \cdot \varepsilon_1) \\
& \left. \left((p_2 \cdot V \cdot H_3)^{a_0 a_1} \mathcal{J}_{15} - (p_2 \cdot N \cdot H_3)^{a_0 a_1} \mathcal{J}_{16} - 4(p_3 \cdot V \cdot H_3)^{a_0 a_1} \mathcal{J}_4 \right) \right] + (2 \leftrightarrow 3) \quad (26)
\end{aligned}$$

where $\mathcal{E}^n = (p_2 \cdot N \cdot \varepsilon_1)^n$. The notation for the appearance of two different consecutive projection operators in the structure $(\mathcal{E} \cdot N \cdot H_3 \cdot V \cdot N \cdot H_2)^{a_1}$, for example, indicate that one transverse leg and one world volume leg of H_2 contract with H_3 and another leg carries the external index a_1 in which contract with the volume $p + 1$ -form tensor. (For another example, in the first structure in the sixth line, three transverse legs of H_2 contract with p_3 , H_3 and ε_1). N and \mathcal{N} were defined to indicate different transvers contractions.

From the contact term contributions in the low energy expansion of the integrals in the above amplitude, one can find the contact amplitude. The Lagrangian corresponding to this contact amplitude becomes

$$\begin{aligned}
\mathcal{L} \sim & \frac{(\pi^2 \alpha')^2 T_1}{6} \epsilon_{a_0 a_1} (C^{(p+1)})_{ij} \\
& \times \left[\frac{1}{2} H^{nij,m} H^{ma_0 a_1, n} - \frac{1}{2} H^{nij,a} H^{aa_0 a_1, n} - H^{nij} H^{aa_0 a_1, an} \right. \\
& + \frac{1}{2} H^{aij,b} H^{ba_0 a_1, a} + \frac{1}{2} H^{aij,n} H^{na_0 a_1, a} - H^{aij} H^{ba_0 a_1, ab} \\
& + H^{aba_1, i} H^{abj, a_0} - 2 H^{anj, a_0} H^{ana_1, i} + H^{nmj, a_0} H^{nma_1, i} \\
& + 2 H^{ana_1, a} H^{nij, a_0} + 2 H^{aba_1, a} H^{bij, a_0} - H^{nma_1} H^{mij, na_0} \\
& + 4 H^{nmi, a_0} H^{mja_1, n} - H^{nmj} H^{ma_0 a_1, in} - H^{naa_1} H^{aij, na_0} \\
& - H^{naj} H^{aa_0 a_1, ni} + H^{ana_1} H^{nij, aa_0} - H^{anj} H^{na_0 a_1, ai} \\
& - H^{aba_1} H^{bij, aa_0} - 4 H^{abi, a_0} H^{bja_1, a} + H^{abj} H^{a_0 a_1 b, ai} \\
& \left. + \frac{1}{2} H^{ij\mu} H^{a_0 a_1 \mu, aa} \right] + (2 \leftrightarrow 3) \tag{27}
\end{aligned}$$

The first term in the fifth line and the second term in the seventh line are exactly the first terms in the first and second lines of the coupling (4) respectively. All other terms are new couplings which should be added to the D-brane action at order α'^2 .

We have found the couplings $C^{(p+1)} \wedge H \wedge H$ by using the explicit calculation of relevant scattering amplitude. Comparing the string low energy amplitude with the Field theory amplitude, one can find all nonzero coupling for two graviton in this case. On the other hand, when one of the indices of the B-field polarization tensors which is contracted with the volume form is the Killing coordinate, the above couplings are not invariant under the linear T-duality. In that case, one should add new couplings involving higher RR potential to make a complete T-dual multiplet. It would be interesting to find this multiplet.

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